

Differential geometry of general affine plane curves*

Zhao Xu-an, zhaoxa@bnu.edu.cn

Gao Hongzhu, hzgao@bnu.edu.cn

Department of Mathematics, Beijing Normal University

Key Laboratory of Mathematics and Complex Systems

Ministry of Education, China, Beijing 100875

Abstract

In this paper we study the general affine geometry of curves in affine space A^2 . For a regular plane curves we define two kinds of moving frames. The first is of minimal order in all moving frames. The second is the Frenet moving frame. We get the moving equations of these moving frames. And we prove that curvature and signature are the complete invariants of regular curves. As application we give a complete classification of constant curvature curves in A^2 .

Keywords: General affine differential geometry, Plane curve, Moving frame, Invariant arc element, Curvature.

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1 Introduction

The affine geometry was founded by Blaschke, Pick, Radon, Berwald and Thomsen among others in the period from 1916 to 1923. And a systematic theory of curves and surfaces in three dimension affine space was developed. For accounts and expository books appeared on the subject, see Blaschke[1], Guggenheimer[5] and Spivak's[8]. In these works the affine geometry means the equiaffine geometry. That is the Kleinnian geometry of the group of affine transformations which preserve volume. But for general affine geometry, that is the Kleinian geometry of the group of general affine transformations, there is little work. All the work we can find are Weise[11][12], Kllingenberg[6][7], Svec[9], Wilkinson[13] and Weiner[10]. Weise and Klingenberg began the study of general affine differential geometry. Wilkinson discussed submanifolds of low codimension. Svec and Weiner studied surfaces in affine space A^4 independently. All the discussions in these papers were in an abstract form. We need a theory of general affine geometry which is parallel to the classical Euclidean geometry of curves and surfaces. And we haven't found any reference on the explicit computation even for curves in A^2 . This is the motivation we write this paper.

The content of this paper is as follows. In section 2 we give an introduction to jet spaces and Fels and Olver's moving frame method. In section 3 we construct a moving frame of minimal order. In section 4 we compute the arc element and the curvature of a regular curve and get the moving equations of the moving frame. In section 5 we construct the Frenet moving frame. In section 6 we give an affine classification curves of constant curvature. In section 7 we discuss modular invariants.

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2 Jet spaces and the moving frame method

The local differential geometry of a curve C at a point P is determined by the shape of C at an arbitrary small neighborhood. So all the local differential geometric properties and invariants of C are determined by the local data of C . It is useful to isolate the informations of curve C at P . This idea hints the concept of jet of curve. The general definition of jet of submanifolds was given by Ehresmann at 1950s for an ambient manifold M .

For a smooth manifold M , let $SM_P^d(M)$ be the set of all smooth d dimensional submanifolds of M that contain the point P . For integer $r \geq 0$, define an equivalence relation \sim on $SM_P^d(M)$ such that $N_1 \sim N_2$ iff N_1 and N_2 have contact at least of order r . The Jet space of d -dimensional submanifolds of M at the point P of order r is the quotient set $J_P^{d,r}(M) = SM_P^d(M)/\sim$. And $J^{d,r}(M) = \bigcup_{P \in M} J_P^{d,r}(M)$ is the jet space of d -dimensional submanifolds of M of order r . An element of jet space $J^{d,r}(M)$ is called a d -jet of order r . For an n dimensional manifold M , $J^{d,r}(M)$ is also a manifold, and the dimension of jet space $J^{d,r}(M)$ is $d + (n - d) \binom{d+r}{r}$. In this paper we use the Jet space $J^{1,r}(A^2)$ for $r > 0$ in the study of curves in A^2 .

2.1 Action of $Aff(2)$ on $J^{1,r}(A^2)$

Let $Aff(2)$ be the group of general affine transformations of affine plane A^2 . The general affine geometry is the Kleinian geometry given by the group action $Aff(2) \times A^2 \rightarrow A^2$. Under the affine coordinates (x, y) on A^2 , a general affine transformation has the form

$$\begin{pmatrix} x' \\ y' \end{pmatrix} = \begin{pmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix} + \begin{pmatrix} x_0 \\ y_0 \end{pmatrix}.$$

Or

$$\begin{pmatrix} x' \\ y' \\ 1 \end{pmatrix} = \begin{pmatrix} a_{11} & a_{12} & x_0 \\ a_{21} & a_{22} & y_0 \\ 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} x \\ y \\ 1 \end{pmatrix}.$$

The coordinates on $Aff(2)$ are $a_{11}, a_{12}, a_{21}, a_{22}, x_0, y_0$.

In this paper, we assume a curve has the form $y = f(x)$ locally. The local coordinates of the manifold $J^{1,r}(M)$ are $x, y, y_x, y_{xx}, \dots, y_{xx \dots x}$, where x appears r times in the subindex of the last item. The action of $Aff(2)$ on A^2 induces an action on $SM^1(A^2)$ and hence on $J^{1,r}(A^2)$, $r \geq 0$.

2.2 Differential invariants of jet spaces

By using jet space as a tool, we can transform the study of local affine congruence invariants of C at P to the invariants of jet space under the action of $Aff(2)$. We have the following definition.

Definition 2.1. A smooth function $f : J^{1,r}(A^2) \rightarrow \mathbb{R}$ which is invariant under the action of $Aff(2)$ is called a differential invariant of curves in A^2 of order less than $r + 1$.

Jet space can be regarded as the finite dimension cut-off of infinite dimension space of all submanifolds. The use of jet space separates the study of differential geometry into algebra part and analysis part. So it makes the structure of differential geometry theory clearer.

2.3 Moving frames for jet spaces of curves

The basic language of modern differential geometry is Cartan's moving frame. For a curve C , the moving frame method gives an affine frame at each point $P \in C$. That is the point $P \in A^2$ and two linearly independent vectors e_1 and e_2 in the associated vector space of A^2 . The moving equations of moving frame are given by

$$dP = w_1 e_1 + w_2 e_2$$

$$d \begin{pmatrix} e_1 \\ e_2 \end{pmatrix} = \begin{pmatrix} w_{11} & w_{12} \\ w_{21} & w_{22} \end{pmatrix} \begin{pmatrix} e_1 \\ e_2 \end{pmatrix}.$$

By Cartan's result, if we construct a moving frame on curve C and compute the one forms $w_1, w_2, w_{11}, w_{12}, w_{21}, w_{22}$, then we determine the congruence class of the curve C .

2.4 Fels and Olver's moving frame method

This section is an introduction to Fels and Olver's moving frame method. See Fels and Olver[2][3][4] for details.

Definition 2.2. Let $\phi : G \times M \rightarrow M$ be a smooth action of Lie group G on smooth manifold M . A moving frame on M is a smooth, G -equivariant map $\rho : M \rightarrow G$.

There are two types of moving frames.

$$\begin{cases} \rho(gz) = g\rho(z), & \text{left moving frame;} \\ \rho(gz) = \rho(z)g^{-1}, & \text{right moving frame.} \end{cases}$$

Theorem 2.1. A moving frame exists on M if and only if G acts freely and regularly on M .

The explicit construction of a moving frame is based on Cartan's normalization procedure. Let G act freely and regularly on M and K be a cross-section to the group orbits, that is a submanifold K which transversally intersects each orbit once. Let g be the unique group element which maps P into the cross-section K , then $\rho : M \rightarrow G, P \mapsto g$ is a right moving frame. And $\rho : M \rightarrow G, P \mapsto g^{-1}$ is a left moving frame. The unique intersection point of the orbit of P and K can be viewed as the canonical form or normal form of P , as prescribed by the cross-section K .

If a moving frame is in hand, the determination of the invariants is routine. The specification of a moving frame by choosing a cross-section induces a canonical procedure to map functions to invariants.

Definition 2.3. The invariantization of a function $F : M \rightarrow \mathbb{R}$ is the unique invariant function $\iota(F)$ that coincides with F on the cross-section, that is $\iota(F)|_K = F|_K$.

Invariantization defines a projection from the space of (smooth) functions to the space of invariants that, moreover, preserves all algebraic operations.

3 The construction of the moving frame

By choosing suitable affine coordinates we can write a curve C as $y = y(x)$ locally.

Given a jet in $J^{1,r}(A^2)$, we use a transportation to move this jet to a jet at $O(0,0)$. The element of $Aff(2)$ fixing the point $(0,0)$ has the form of $x' = a_{11}x + a_{12}y, y' = a_{21}x + a_{22}y$.

By direct computation, we get the action of $Aff(2)$ on $J^{1,5}(A^2)$. The explicit formulae are given by

$$y'_{x'} = \frac{a_{21} + a_{22}y_x}{a_{11} + a_{12}y_x} \quad (1)$$

$$y'_{x'x'} = \frac{\Delta y_{xx}}{\Gamma^3} \quad (2)$$

$$y'_{x'x'x'} = \frac{\Delta(a_{11}y_{xxx} + a_{12}y_x y_{xxx} - 3a_{12}y_{xx}^2)}{\Gamma^5} \quad (3)$$

$$y'_{x'x'x'x'} = \frac{\Delta M}{\Gamma^7} \quad (4)$$

$$y'_{x'x'x'x'x'} = \frac{\Delta N}{\Gamma^9} \quad (5)$$

Where $\Delta = a_{11}a_{22} - a_{21}a_{12}$, $\Gamma = a_{11} + a_{12}y_x$,

$$M = \Gamma^2 y_{xxxx} - 10a_{12}\Gamma y_{xx}y_{xxx} + 15a_{12}^2\Gamma y_{xx}^3,$$

$$N = \Gamma^3 y_{xxxxx} - 15a_{12}\Gamma^2 y_{xx}y_{xxxx} - 10a_{12}\Gamma^2 y_{xx}^2 + 105a_{12}^2\Gamma y_{xx}^2 y_{xxx} - 105a_{12}^3 y_{xx}^4.$$

3.1 Explicit construction of moving frame

In this section we use Fels and Olver's moving frame method to construct a right moving frame on jet space $J^{1,4}(A^2)$ and simultaneously we show how to transform a jet of curve at $P(x,y)$ into a standard jet at $(0,0)$.

1. Since $y'_{x'} = \frac{a_{21} + a_{22}y_x}{a_{11} + a_{12}y_x}$, if we fix $y'_{x'} = 0$, then we get

$$a_{21} + a_{22}y_x = 0 \quad (6)$$

and $a_{11} + a_{12}y_x \neq 0$. So we have

$$y_x = -\frac{a_{21}}{a_{22}} \quad (7)$$

Here we require $a_{22} \neq 0$. Otherwise $a_{21} = -a_{22}y_x = 0$, as a consequence $\Delta = 0$. This is a contradiction.

Under this condition we have $\Gamma = a_{11} + a_{12}y_x = a_{11} + a_{12}\frac{-a_{21}}{a_{22}} = \frac{\Delta}{a_{22}}$.

2. Since $y'_{x'x'} = \frac{\Delta y_{xx}}{\Gamma^3}$, if we assume $y_{xx} \neq 0$, then it is reasonable to set $y'_{x'x'} = 1$. Substituting $\Gamma = \frac{\Delta}{a_{22}}$ into $y'_{x'x'} = \frac{\Delta y_{xx}}{\Gamma^3}$, we get

$$y_{xx}a_{22}^3 = \Delta^2. \quad (8)$$

Hence

$$y_{xx} = \frac{\Delta^2}{a_{22}^3}. \quad (9)$$

This require that y_{xx} has the same sign with a_{22} .

3. We substitute the expressions of y_x and y_{xx} in Equation 7 and 9 into Equation 3 and get

$$y'_{x'x'x'} = a_{22}^4 \Delta^{-3} (y_{xxx} - \frac{3a_{12}\Delta^3}{a_{22}^5}).$$

We fix $y'_{x'x'x'} = 0$. That is

$$y_{xxx} = \frac{3a_{12}\Delta^3}{a_{22}^5} \quad (10)$$

Hence

$$a_{12} = \frac{1}{3} a_{22}^5 \Delta^{-3} y_{xxx} \quad (11)$$

4. As in 3, we substitute the expressions of y_x, y_{xx} and y_{xxx} in Equation 7, 9 and 10 into Equation 4 and get

$$y'_{x'x'x'x'} = a_{22}^5 \Delta^{-4} (y_{xxxx} - \frac{15a_{12}^2 \Delta^4}{a_{22}^7}) = \frac{1}{a_{22} y_{xx}^2} (y_{xxxx} - \frac{5y_{xxx}^2}{3y_{xx}})$$

Since we have assumed $y_{xx} \neq 0$, and y_{xx} has the same sign with a_{22} , if we further assume $y_{xxxx} - \frac{5y_{xxx}^2}{3y_{xx}} \neq 0$, $y'_{x'x'x'x'}$ must have the same sign with $y_{xx}(y_{xxxx} - \frac{5y_{xxx}^2}{3y_{xx}}) = \frac{1}{3}(3y_{xx}y_{xxxx} - 5y_{xxx}^2)$.

We fix $y'_{x'x'x'x'} = \begin{cases} 1, & \text{if } 3y_{xx}y_{xxxx} - 5y_{xxx}^2 > 0; \\ -1, & \text{if } 3y_{xx}y_{xxxx} - 5y_{xxx}^2 < 0. \end{cases}$

In either cases we have

$$a_{22} = \frac{1}{3y_{xx}^3} |3y_{xx}y_{xxxx} - 5y_{xxx}^2| \quad (12)$$

In the previous four steps we transform the jet of curve C at $P(x, y)$ into a jet with $x' = y' = y'_{x'} = y'_{x'x'} = 0$, $y'_{x'x'} = 1$ and $y'_{x'x'x'} = \pm 1$ if both y_{xx} and $3y_{xx}y_{xxxx} - 5y_{xxx}^2$ are not 0. In the following we compute the affine transformation to get the moving frame.

From Equation 6, we have

$$a_{21} = -\frac{y_x}{3y_{xx}^3} |3y_{xx}y_{xxxx} - 5y_{xxx}^2| \quad (13)$$

If we assume $\Delta > 0$, then from Equation 8 and 12 we have

$$\Delta = \sqrt{y_{xx} a_{22}^3} = \sqrt{\frac{1}{27y_{xx}^8} |3y_{xx}y_{xxxx} - 5y_{xxx}^2|^3} \quad (14)$$

Substituting Δ into Equation 11, we get

$$a_{12} = \frac{1}{3\sqrt{3}} \frac{y_{xxx} |3y_{xx}y_{xxxx} - 5y_{xxx}^2|^{\frac{1}{2}}}{y_{xx}^3} \quad (15)$$

By virtue of $a_{11}a_{22} - a_{21}a_{12} = \Delta$, we have

$$a_{11} = -\frac{1}{3\sqrt{3}} \frac{(y_x y_{xxx} - 3y_{xx}^2)|3y_{xx}y_{xxxx} - 5y_{xxx}^2|^{\frac{1}{2}}}{y_{xx}^3} \quad (16)$$

Summarizing the results in the previous discussions, and setting $S_1 = y_{xx}$, $S_2 = 3y_{xx}y_{xxxx} - 5y_{xxx}^2$, then we have

Proposition 3.1. *Let j be a jet of curve $y = y(x)$ at (x_0, y_0) with $S_1 \neq 0$ and $S_2 \neq 0$, then there exists a unique element $A \in \text{Aff}(2)$ with $\Delta > 0$ such that Aj is a jet of curve $y' = y'(x')$ satisfying $x' = y' = y'_{x'} = y'_{x'x'} = 0, y'_{x'x'} = 1$ and $y'_{x'x'x'} = \pm 1$. Furthermore A has the form*

$$\begin{pmatrix} x' \\ y' \end{pmatrix} = \begin{pmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{pmatrix} \begin{pmatrix} x - x_0 \\ y - y_0 \end{pmatrix}$$

Where $a_{22}, a_{21}, a_{12}, a_{11}$ are given by Equation 12, 13, 15 and 16.

By this proposition, we give the following definition.

Definition 3.1. 1. *Let $C : y = y(x)$ be a smooth curve. A point $P(x, y) \in C$ satisfying $S_1 \neq 0$ and $S_2 \neq 0$ is called a regular point of C . A point which is not regular is called a singular point.*

2. *A jet $j \in J^{1,4}(A^2)$ is regular if there exist affine coordinates x, y such that j has the form $y = y(x)$ locally with both $S_1 \neq 0$ and $S_2 \neq 0$. A jet $j \in J^{1,4}(A^2)$ is singular if it is not regular.*

The definition of regularity of a jet does not depend on the affine coordinates. Direct computation shows under affine transformations, we have $S'_1 = \Delta \Gamma^{-3} S_1$, $S'_2 = \Delta^2 \Gamma^{-8} S_2$.

Proposition 3.2. *If a smooth curve $C : y = y(x)$ is regular, then we have a smooth right moving frame (P, α_1, α_2) on C , where*

$$\begin{pmatrix} \alpha_1 \\ \alpha_2 \end{pmatrix} = \begin{pmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{pmatrix} = \begin{pmatrix} \frac{-\sqrt{3}(y_x y_{xxx} - 3y_{xx}^2)|3y_{xx}y_{xxxx} - 5y_{xxx}^2|^{\frac{1}{2}}}{9y_{xx}^3} & \frac{\sqrt{3}y_{xxx}|3y_{xx}y_{xxxx} - 5y_{xxx}^2|^{\frac{1}{2}}}{9y_{xx}^3} \\ -\frac{y_x}{3y_{xx}^3}|3y_{xx}y_{xxxx} - 5y_{xxx}^2| & \frac{1}{3y_{xx}^3}|3y_{xx}y_{xxxx} - 5y_{xxx}^2| \end{pmatrix}$$

Proposition 3.3. *If all the points of curve C is singular, then there exists C_0, C_1, C_2, C_3 such that $y = (C_3x + C_2)^{\frac{1}{2}} + C_1x + C_0$ or there exists C_0, C_1, C_2 such that $y = C_2x^2 + C_1x + C_0$. Equivalently C is affine congruence to parabola $y = x^2$ or line $y = x$.*

Proof: The singular condition is $3y_{xxxx}y_{xx} - 5y_{xxx}^2 = 0$. It can be written as $\frac{3y_{xxxx}}{y_{xxx}} = \frac{5y_{xxx}}{y_{xx}}$, if $y_{xxx} \neq 0$, we have $3 \ln |y_{xxx}| = 5 \ln |y_{xx}|$. Solving this differential equation will finish the proof.

4 Invariant arc element, curvature and the moving equations of moving frame

In this section, we derive the invariant arc element and curvature for an affine plane curve.

Proposition 4.1. *The affine invariant arc element is $ds = \frac{|3y_{xx}y_{xxxx} - 5y_{xxx}^2|^{\frac{1}{2}}}{\sqrt{3}y_{xx}}dx$. As a consequence $\frac{d}{ds} = \frac{\sqrt{3}y_{xx}}{|3y_{xx}y_{xxxx} - 5y_{xxx}^2|^{\frac{1}{2}}} \frac{d}{dx}$ is an affine invariant differential operator.*

Proof: We use the right moving frames to compute the affine invariant arc element. Differentiate $x' = a_{11}x + a_{12}y$, we get $dx' = a_{11}dx + a_{12}dy$. On the curve C , we have $dy = y_x dx$. By the Normalization procedure, we substitute dy and a_{11}, a_{21} into the expression of dx' and get $dx' = \frac{|3y_{xx}y_{xxxx} - 5y_{xxx}^2|^{\frac{1}{2}}}{\sqrt{3}y_{xx}}dx$. Since dx' is the invariantization of dx , it can be chosen as the invariant arc element. Dually $\frac{d}{dx'}$ is an invariant differential operator.

Theorem 4.1. *The curvature of curve $C : y = y(x)$ at the point $P(x, y)$ is*

$$k(x) = \frac{\sqrt{3}(9y_{xx}^2y_{xxxxx} - 45y_{xx}y_{xxx}y_{xxxx} + 40y_{xxx}^3)}{3|3y_{xx}y_{xxxx} - 5y_{xxx}^2|^{\frac{3}{2}}}.$$

Proof: Substituting Equations 12,13,15,16 into Equation 5, we get the invariantization of y_{xxxxx} which is the curvature.

Theorem 4.2. *The moving equation of the right moving frame is*

$$\frac{d}{ds} \begin{pmatrix} \alpha_1 \\ \alpha_2 \end{pmatrix} = \begin{pmatrix} \frac{1}{2}k(s) & \frac{1}{3}\sigma \\ -1 & k(s) \end{pmatrix} \begin{pmatrix} \alpha_1 \\ \alpha_2 \end{pmatrix}$$

Where $\sigma = \text{sgn}(S_2)$, $\alpha_1 = \left(\frac{-\sqrt{3}(y_x y_{xxx} - 3y_{xx}^2)|3y_{xx}y_{xxxx} - 5y_{xxx}^2|^{\frac{1}{2}}}{9y_{xx}^3}, \frac{\sqrt{3}y_{xxx}|3y_{xx}y_{xxxx} - 5y_{xxx}^2|^{\frac{1}{2}}}{9y_{xx}^3} \right)$

and

$$\alpha_2 = \left(-\frac{y_x}{3y_{xx}^3}|3y_{xx}y_{xxxx} - 5y_{xxx}^2|, \frac{1}{3y_{xx}^3}|3y_{xx}y_{xxxx} - 5y_{xxx}^2| \right)$$

Proof: This is derived by direct computation from the results in Proposition 3.2, Proposition 4.1 and Theorem 4.1.

The left moving frame is given by $(e_1, e_2) = \begin{pmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{pmatrix}^{-1}$.

Theorem 4.3. *The moving equation of the left moving frame is*

$$\frac{d}{ds}(e_1, e_2) = (e_1, e_2) \begin{pmatrix} -\frac{1}{2}k(s) & -\frac{1}{3}\sigma \\ 1 & -k(s) \end{pmatrix}$$

$$\text{Where } e_1 = \begin{pmatrix} \frac{\sqrt{3}y_{xx}}{|3y_{xx}y_{xxxx} - 5y_{xxx}^2|^{\frac{1}{2}}} \\ \frac{\sqrt{3}y_x y_{xx}}{|3y_{xx}y_{xxxx} - 5y_{xxx}^2|^{\frac{1}{2}}} \end{pmatrix}, e_2 = \begin{pmatrix} -\frac{y_{xx}y_{xxxx}}{|3y_{xx}y_{xxxx} - 5y_{xxx}^2|^{\frac{1}{2}}} \\ -\frac{y_{xx}(y_x y_{xxx} - 3y_{xx}^2)}{|3y_{xx}y_{xxxx} - 5y_{xxx}^2|^{\frac{1}{2}}} \end{pmatrix}$$

Proof: Let $A = \begin{pmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{pmatrix}$ and $K = \begin{pmatrix} \frac{1}{2}k(s) & \frac{1}{3}\sigma \\ -1 & k(s) \end{pmatrix}$. We have $dA^{-1} = -A^{-1}dAA^{-1} = -A^{-1}KAA^{-1} = -A^{-1}K$.

By Cartan's moving frame method, Theorem 4.2 or Theorem 4.3 gives the general affine classification of regular plane curves.

Corollary 4.1. *For a regular smooth plane curve, $k = k(s)$ and σ are the complete affine invariants. All the differential invariants are functions of $k(s)$ and its derivatives on s .*

Remark 4.1. *The discuss in this section is for the geometry of orientation preserving affine transformations. If a transformation reverses the orientation, then it changes the sign of $k(s)$ and keeps σ invariant.*

5 The Frenet moving frame

The left moving frame e_1, e_2 we constructed in the last section uses the derivatives of $y(x)$ up to order 4 and the moving equations of the moving frame contains derivatives up to order 5.

Using arc element we have a left moving frame which contains the derivatives of $y(x)$ up to order 5, so the moving equations of moving frame contains derivatives of order 6. We call it Frenet frame. Computation gives the following result.

Theorem 5.1. *Let $r = \begin{pmatrix} x \\ y(x) \end{pmatrix}$, $t = \frac{dr}{ds}$ and $n = \frac{dt}{ds}$, then*

$$\frac{d}{ds}(t, n, r) = (t, n, r) \begin{pmatrix} 0 & -\frac{1}{2}k^2 - \frac{1}{2}\frac{dk}{ds} - \frac{\sigma}{3} & 1 \\ 1 & -\frac{3}{2}k & 0 \\ 0 & 0 & 0 \end{pmatrix}$$

Proof: Let $r = \begin{pmatrix} x \\ y(x) \end{pmatrix}$, computation shows $t = \frac{dr}{ds} = e_1$. We set $\frac{dt}{ds} = n$. So $n = \frac{de_1}{ds} = -\frac{1}{2}k(s)t + e_2$ and $\frac{dn}{ds} = (-\frac{1}{2}k^2 - \frac{1}{2}\frac{dk}{ds} - \frac{\sigma}{3})t - \frac{3}{2}kn$.

The moving frame e_1, e_2 is valid for curve with derivative up to order 5, but the Frenet frame t, n is only valid for curve with derivative up to order 6.

Definition 5.1. *The vectors t and n are respectively called the affine tangential vector and normal vector of the curve C .*

6 Curves with constant curvature

In this section we give the classification of curves with constant curvature in A^2 . We assume all curves are regular. The one parameter subgroup of $Aff(2)$ has the form $\exp(tX)$, where $X \in aff(2)$ and $aff(2)$ is the Lie algebra of $Aff(2)$. We write $A \in Aff(2)$ as a 3×3 matrix $\begin{pmatrix} a_{11} & a_{12} & x_0 \\ a_{21} & a_{22} & y_0 \\ 0 & 0 & 1 \end{pmatrix}$.

The following proposition gives the one parameter subgroups of $Aff(2)$ up to conjugation.

Proposition 6.1. *Let $G = \exp(tX)$ be a one parameter subgroup of $Aff(2)$,*

1. *If G fixes a point $P \in A^2$, then up to conjugation X has the form of*

$$\begin{pmatrix} \lambda_1 & 0 & 0 \\ 0 & \lambda_2 & 0 \\ 0 & 0 & 0 \end{pmatrix}, \begin{pmatrix} \lambda & 1 & 0 \\ 0 & \lambda & 0 \\ 0 & 0 & 0 \end{pmatrix} \text{ or } \begin{pmatrix} \lambda & -\mu & 0 \\ \mu & \lambda & 0 \\ 0 & 0 & 0 \end{pmatrix}.$$

2. If G does not fix any point in A^2 , then up to conjugation X has the form of

$$\begin{pmatrix} 0 & 0 & a \\ 0 & \lambda & 0 \\ 0 & 0 & 0 \end{pmatrix} \text{ or } \begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{pmatrix}$$

Proof: If G fixes a point $P \in A^2$, then $\text{Exp}(tX)P = P, \forall t \in \mathbb{R}$. By choosing P as origin, we have $\text{Exp}(tX)$ fix the origin. So X must have the form listed in 1.

If G does not fix any point in A^2 , then X will have an eigenvalue 0. Computation shows that X must have the form listed in 2.

Corollary 6.1. *The one parameter subgroup corresponding to X is given by*

$$1. \begin{pmatrix} e^{\lambda_1 t} & 0 & 0 \\ 0 & e^{\lambda_2 t} & 0 \\ 0 & 0 & 1 \end{pmatrix}, \begin{pmatrix} e^{\lambda t} & te^{\lambda t} & 0 \\ 0 & e^{\lambda t} & 0 \\ 0 & 0 & 1 \end{pmatrix} \text{ or } \begin{pmatrix} e^{\lambda t} \cos(\mu t) & -e^{\lambda t} \sin(\mu t) & 0 \\ e^{\lambda t} \sin(\mu t) & e^{\lambda t} \cos(\mu t) & 0 \\ 0 & 0 & 1 \end{pmatrix}.$$

$$2. \begin{pmatrix} 1 & 0 & at \\ 0 & e^{\lambda t} & a(e^{\lambda t} - 1) \\ 0 & 0 & 1 \end{pmatrix} \text{ or } \begin{pmatrix} 1 & t & \frac{1}{2}t^2 \\ 0 & 1 & t \\ 0 & 0 & 1 \end{pmatrix}$$

Given a point $P(x_0, y_0) \in A^2$, the curve $\text{Exp}(tX)P$ has constant curvature.

Theorem 6.1. *Up to affine congruence, the curves generated by the action of one parameter subgroups of $\text{Aff}(2)$ on A^2 have the forms*

1. $y = x^a, a \neq 0$.
2. $y = ax + bx \ln |x|$;
3. $\sqrt{(x^2 + y^2)} = e^{\frac{a}{b} \arctan(\frac{y}{x})}$. In polar coordinates it is $r = e^{\frac{a}{b}\theta}$. If $a = 0$, then this is a circle.
4. $y = e^x$;

Remark 6.1. *The curve $y = x^a$ is affine congruence to the curve $y = x^{\frac{1}{a}}$, and the curve $y = ax + bx \ln |x|$ is affine congruence to the curve $y = x \ln(x)$; The first three cases of Theorem 6.1 correspond to the three cases in 1 of Corollary 6.1. The fourth case of Theorem 6.1 corresponds to the first case in 2 of Corollary 6.1 with $\lambda \neq 0$. The case with $\lambda = 0$ and the second case in 2 of Corollary 6.1 give line and parabola respectively, and they are contained in the first case in Theorem 6.1.*

Theorem 6.2. *If C has constant curvature, then C is affine congruence to one of the curve in Theorem 6.1.*

Proof: We assume C is a smooth curve with constant curvature k , in fact C has continuous derivative of order 6 is enough. Let $F = \begin{pmatrix} t(s) & n(s) & r(s) \\ 0 & 0 & 1 \end{pmatrix}$. By Theorem 5.1 the moving equa-

tion of C can be written as $\frac{dF(s)}{ds} = FX$, where $X = \begin{pmatrix} 0 & -\frac{1}{2}k^2 - \frac{1}{2}\frac{dk}{ds} - \frac{\sigma}{3} & 1 \\ 1 & -\frac{3}{2}k & 0 \\ 0 & 0 & 0 \end{pmatrix}$. Its solution

is $F(s) = F(0)Exp(sX)$. By taking the third column of F we get $\begin{pmatrix} r(s) \\ 1 \end{pmatrix} = F(s) \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix} = F(0)Exp(sX) \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix}$. Hence C is affine congruence to a curve listed in Theorem 6.1.

Theorem 6.3. *The invariants of the curves with constant curvature are given in the following.*

1. For $y = x^a$, $k(x) = \frac{2\sqrt{3}}{3} \operatorname{sgn}(xa(a-1)(a-\frac{1}{2})(a-2)) \frac{(a+1)}{\sqrt{|(2a-1)(a-2)|}}$. For $a = 0, \frac{1}{2}, 1, 2$, $S_2 = 0$ and k is not well defined. If $\frac{1}{2} < a < 2$ then $\sigma > 0$, otherwise $\sigma < 0$.
2. For $y = bx + ax \ln |x|$, $k(x) = -\frac{4\sqrt{3}}{3} \operatorname{sgn}(ax)$ and $\sigma > 0$.
3. For $\sqrt{(x^2 + y^2)} = e^{\frac{a}{b} \arctan(\frac{y}{x})}$, $k(x) = -\frac{4\sqrt{3}}{3} \frac{a \operatorname{sgn}(b)}{(a^2 + 9b^2)^{\frac{1}{2}}}$ and $\sigma > 0$.
4. For $y = e^x$, $k(x) = \frac{\sqrt{6}}{3}$ and $\sigma < 0$.

Proof: Direct computation.

Corollary 6.2. *If C is a regular curves with zero curvature, then C is part of an eclipse or a hyperbola.*

7 Modular invariants

We have defined expression S_1, S_2 before. Let $S_3 = -45y_{xx}y_{xxx}y_{xxxx} + 9y_{xx}^2y_{xxxxx} + 40y_{xxx}^3$, then direct computation shows

$$\begin{aligned} S'_1 &= \Delta \Gamma^{-3} S_1; \\ S'_2 &= \Delta^2 \Gamma^{-8} S_2; \\ S'_3 &= \Delta^3 \Gamma^{-12} S_3. \end{aligned}$$

We give the following definition.

Definition 7.1. *Let $f(y, y_x, y_{xx}, \dots)$ be a rational expression of $y(x)$ and its derivatives with x , we call f a modular invariant of weight (p, q) if under any affine transformation $f(y', y'_{x'}, y'_{x'x'}, \dots) = \Delta^p \Gamma^q f(y, y_x, y_{xx}, \dots)$.*

By this definition, S_1, S_2, S_3 are modular invariants of weight $(1, -3), (2, -8), (3, -12)$. It is obvious that a modular invariant of weight $(0, 0)$ is a differential invariant. All the modular invariants of weight (p, q) form a real vector space $M_{p,q}$. $M = \bigoplus_{p,q \in \mathbb{Z}} M_{p,q}$ is a double graded real algebra.

The following proposition determine the structure of $M_{p,q}$.

Proposition 7.1. 1. $T_1 = \frac{S_1^4}{S_3}$ is a modular invariant of weight $(1, 0)$ and $T_2 = \frac{S_1 S_2}{S_3}$ is a modular invariant of weight $(0, 1)$. 2. $M_{p,q} = T_1^p T_2^q M_{0,0}$.

For an element $f \in M_{p,q}$, $f = 0$ gives an affine invariant differential operator.

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